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On the Korteweg-de Vries Equation: Existence and Uniqueness*

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1. INTRODUCTION

There has been substantial recent interest in the Korteweg-de Vries equation

$$u_t = uu_x + \delta u_{xxx}, \qquad \delta \neq 0, \tag{1.1}$$

as evidenced by the work of Kruskal [2], Whitham [6], Lax [3], Zabusky [7], Miura [4], Miura *et al.* [5], and Gardner *et al.* [1], among others.

In this paper we shall produce an existence and uniqueness theorem for the problem defined by (1.1), the initial condition

$$u(x, 0) = f(x), \qquad \text{all } x \tag{1.2}$$

and the boundary condition

$$u(x, t) = u(x + 1, t), \qquad \text{all } x \text{ and } t. \tag{1.3}$$

We shall refer to this problem as problem (1) and we shall prove:

THEOREM 1. *If* $\delta \neq 0$ *and if* $f(x)$ *is a* 1-*periodic function, whose third derivative belongs to the space* L_2, *then there exists a unique solution of problem* (1).

If we instead consider the Cauchy problem for (1.1) we can prove an analogous theorem. This is true because we use only L_2 estimates in the proof of Theorem 1.

The proof consists of the following three steps:

(i) Let N be a natural number, $h = 1/N$ and $x_r = rh$. Denote by D_+, D_-, and D_0 the difference operators defined by

$$hD_+g(x_r) = g(x_{r+1}) - g(x_r), \qquad hD_-g(x_r) = g(x_r) - g(x_{r-1}),$$
$$2hD_0g(x_r) = g(x_{r+1}) - g(x_{r-1}).$$

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Instead of problem (1) consider another problem, which we call problem (2), and which is given by the differential-difference equation

$$\partial u_N(x_r, t)/\partial t = [u_N(x_r, t)\, D_0 u_N(x_r, t) + D_0 u_N(x_r, t)^2]/3 + \delta D_+ D_-^2 u_N(x_r, t), \qquad r = 1, 2, \ldots, N \tag{1.4}$$

the initial condition

$$u_N(x_r, 0) = f(x_r), \qquad r = 1, 2, \ldots, N \tag{1.5}$$

and the boundary condition

$$u_N(x_r, t) = u_N(x_{r+N}, t) \qquad \text{all} \quad r \quad \text{and} \quad t. \tag{1.6}$$

Problem (2) has for every $h > 0$ a unique solution $u_N(x, t)$. This is clear if we take into account the bound (3.1) derived in section 3. The solution $u_N(x, t)$ is for every t a gridfunction, i.e. defined only for $x = x_r = rh$. The derivatives $\partial^n u_N(x_r, t)/\partial t^n$, $n > 0$, all exist.

(ii) The next step is to prove the existence of a *local* solution of problem (1), i.e. a solution in an interval $0 \leqslant t \leqslant T_1$. For this purpose we derive bounds for $u_N(x, t)$ and its differences of order three and lower. These bounds are uniform in N and hold for some interval $0 \leqslant t \leqslant T_1$.

We then consider the discrete Fourier series $\phi_N(x, t)$ corresponding to $u_N(x, t)$. This series is defined for all x and interpolates $u_N(x, t)$ at $x = x_r$. From the bounds for $u_N(x, t)$ and its differences we get bounds for $\phi_N(x, t)$ and its derivatives up to order three. These bounds are also uniform in N and imply that the sequence $\{\phi_N(x, t)\}$ contains a subsequence converging to a local solution of problem (1).

(iii) The last step is to show that the local solution can be extended to a *global* solution. Suppose that $u(x, t)$ is a solution of problem (1). In Section 4 we derive some a priori bounds for $u(x, t)$, and some of its derivatives. As we already have proven existence in $0 \leqslant t \leqslant T_1$, these bounds certainly hold in $0 \leqslant t \leqslant T_1$. It is then possible to consider problem (1) with $u(x, T_1)$ substituted for the initial function $f(x)$. This new problem also has a local solution and hence, we get existence of a solution in $0 \leqslant t \leqslant T_2$, where $T_2 > T_1$. We then show that repeated use of this extension procedure will give existence for all t.

We still must prove the uniqueness part of Theorem 1. This is done in Section 5.

In a forthcoming paper we will discuss a numerical procedure for solving problem (1) and present some numerical results.

2. Technical Lemmas

We begin with a well-known Sobolev-type lemma:

LEMMA 2.1. *Let σ and τ be integers such that $0 \leqslant \tau < \sigma$. Then to every constant $\epsilon > 0$ there exists a constant $c(\epsilon)$ such that for all functions y, sufficiently differentiable on $0 \leqslant x \leqslant 1$,*

$$\max_{0 \leqslant x \leqslant 1} |\partial^\tau y/\partial x^\tau|^2 \leqslant \epsilon \|\partial^\sigma y/\partial x^\sigma\|^2 + c(\epsilon)\|y\|^2 \tag{2.1}$$

and

$$\|\partial^\tau y/\partial x^\tau\|^2 \leqslant \epsilon \|\partial^\sigma y/\partial x^\sigma\|^2 + c(\epsilon)\|y\|^2, \tag{2.2}$$

where

$$\|f\|^2 = (f, f) \text{ and } (f, g) = \int_0^1 \overline{f(x)}\, g(x)\, dx.$$

In the space of gridfunctions we define the scalar product and the norm by

$$(f, g)_h = \sum_{r=1}^{N} \overline{f(x_r)}\, g(x_r)\, h \qquad \text{and} \qquad \|f\|_h^2 = (f, f)_h\,.$$

We shall always assume that N is of the form $N = 2n + 1$ where n is a natural number. It is then easy to see that the functions $\{e^{2\pi i\omega x}\}_{\omega=-n}^{n}$ are orthonormal with respect to the scalar products $(.,.)_h$ and $(.,.)$.

LEMMA 2.2. *Let τ_1 and τ_2 be nonnegative integers with $\tau_1 + \tau_2 = \tau$ and ψ a function of the form*

$$\psi = \sum_{\omega=-n}^{n} a(\omega)\, e^{2\pi i\omega x}.$$

Then

$$\left(\frac{2}{\pi}\right)^{2\tau} \left\| \frac{\partial^\tau \psi}{\partial x^\tau} \right\|^2 \leqslant \| D_+^{\tau_1} D_-^{\tau_2} \psi \|^2 = \| D_+^{\tau_1} D_-^{\tau_2} \psi \|_h^2 \leqslant \left\| \frac{\partial^\tau \psi}{\partial x^\tau} \right\|^2.$$

Proof. We observe that

$$D_+ e^{2\pi i\omega x} = \frac{e^{2\pi i\omega h} - 1}{h}\, e^{2\pi i\omega x}$$

and

$$\left| \frac{e^{2\pi i\omega h} - 1}{h} \right| = \left| \frac{2 \sin \pi\omega h}{h} \right|.$$

By orthonormality we get

$$\| D_+^\tau \psi \|^2 = \| D_+^\tau \psi \|_h^2 = \sum_{\omega=-n}^{n} \left| \frac{2 \sin \pi\omega h}{h} \right|^{2\tau} | a(\omega)|^2$$

and similarly

$$\left\| \frac{\partial^\tau \psi}{\partial x^\tau} \right\|^2 = \sum_{\omega=-n}^{n} | 2\pi i\omega |^{2\tau} | a(\omega)|^2.$$

The lemma now follows immediately in the case $\tau_2 = 0$ if we use the elementary inequality

$$\frac{2}{\pi} \leqslant \left| \frac{\sin \pi\omega h}{\pi\omega h} \right| \leqslant 1, \qquad | \omega | \leqslant n. \tag{2.3}$$

For the case $\tau_2 \neq 0$ we use the relations

$$D_- e^{2\pi i\omega x} = \frac{1 - e^{-2\pi i\omega h}}{h} e^{2\pi i\omega x}, \qquad \left| \frac{1 - e^{-2\pi i\omega h}}{h} \right| = \left| \frac{2 \sin \pi\omega h}{h} \right|,$$

and (2.3). We now prove a discrete analogue of (2.2).

LEMMA 2.3. *Let z be an N-periodic gridfunction and let σ and τ be integers such that $0 \leqslant \tau < \sigma$. Then for every constant $\epsilon > 0$ there exists a constant $c(\epsilon)$ independent of z and h such that*

$$\| D_+{}^\tau z \|_h^2 \leqslant \epsilon \| D_+{}^\sigma z \|_h^2 + c(\epsilon) \| z \|_h^2 . \tag{2.4}$$

Proof. Let ψ be the discrete Fourier series of z, i.e.,

$$\psi = \sum_{\omega=-n}^{n} b(\omega)\, e^{2\pi i\omega x},$$

where $b(\omega) = (e^{2\pi i\omega x}, z)_h$.

From Lemma 2.1 and Lemma 2.2 we get

$$\begin{aligned}
\| D_+{}^\tau z \|_h^2 &= \| D_+{}^\tau \psi \|_h^2 \leqslant \left\| \frac{\partial^\tau \psi}{\partial x^\tau} \right\|^2 \\
&\leqslant \epsilon' \left\| \frac{\partial^\sigma \psi}{\partial x^\tau} \right\|^2 + c'(\epsilon') \| \psi \|^2 \\
&\leqslant \epsilon' \left(\frac{\pi}{2}\right)^{2\sigma} \| D_+{}^\sigma \psi \|_h^2 + c'(\epsilon') \| \psi \|_h^2 \\
&= \epsilon' \left(\frac{\pi}{2}\right)^{2\sigma} \| D_+{}^\sigma z \|_h^2 + c'(\epsilon') \| z \|_h^2 = \epsilon \| D_+ z \|_h^2 + c(\epsilon) \| z \|_h^2 ,
\end{aligned}$$

which was to be proven.

Remark 1. The inequality (2.4) can be modified in the following ways. The operator D_+^τ in the left member can be exchanged for any operator $D_+^{\tau_1} D_0^{\tau_2} D_-^{\tau_3}$, where τ_1, τ_2 and τ_3 are nonnegative integers with $\tau_1 + \tau_2 + \tau_3 = \tau$. The operator D_+^σ in the right member can be exchanged for any operator $D_+^{\sigma_1} D_-^{\sigma_2}$, where $\sigma_1 + \sigma_2 = \sigma$ and σ_1, σ_2 are nonnegative integers. It can be proven by counterexamples that the lemma is not true if the right member contains D_0.

Remark 2. Lemma 2.1 and Lemma 2.3 immediately give

$$\max_{1 \leqslant r \leqslant N} | D_0 z_r |^2 \leqslant \epsilon \| D_+ D_-^2 z \|_h^2 + c(\epsilon) \| z \|_h^2 , \tag{2.5}$$

where ϵ and $c(\epsilon)$ have the same properties as in Lemma 2.3. This inequality will be of use in Section 3.

LEMMA 2.4. *If f is a real N-periodic gridfunction*, i.e., if $f(x_r) = f(x_{r+N})$, *and if $g(x)$ is another real N-periodic gridfunction, then*

$$(f, D_+ g)_h = -(D_- f, g)_h ,$$

$$(f, D_- g)_h = -(D_+ f, g)_h ,$$

$$(f, D_0 g)_h = -(D_0 f, g)_h ,$$

and

$$(f, D_+ D_-^2 f)_h \leqslant 0 \tag{2.6}$$

Proof. The first three identities are easily verified by use of the definition of scalar product. Hence,

$$\begin{aligned} (f, D_+ D_-^2 f)_h &= -(D_+^2 D_- f, f)_h = (f, (D_+ D_-^2 - D_+^2 D_-) f)_h / 2 \\ &= (f, D_+ D_- (D_- - D_+) f)_h / 2. \end{aligned}$$

As $D_- - D_+ = -h D_+ D_-$, we get

$$(f, D_+ D_-^2 f)_h = -h (f, D_+^2 D_-^2 f)_h / 2 = -h \| D_+ D_- f \|_h^2 / 2$$

and (2.6) is established.

In the next lemma we state some a priori properties of the solution of problem (1). The first two were found by Kruskal and Zabusky [7], and the third by Whitham [6]. Further properties of the same type but containing derivatives of higher order can be found in [5].

LEMMA 2.5. *Let $u(x, t)$ be a solution of problem* (1). *Then there are constants* α_1, α_2, α_3 *such that*

$$\int_0^1 u^2(x, t)\, dx = \int_0^1 f^2\, dx = \alpha_1, \tag{2.7}$$

$$\int_0^1 (u^3/3 - \delta u_x^2)\, dx = \int_0^1 (f^3/3 - \delta f'^2)\, dx = \alpha_2, \tag{2.8}$$

$$\begin{aligned}&\int_0^1 (u^4 - 12\delta u u_x^2 + 36\delta^2 u_{xx}^2/5)\, dx\\ &\qquad = \int_0^1 (f^4 - 12\delta f f' + 36\delta^2 f''^2/5)\, dx = \alpha_3 .\end{aligned} \tag{2.9}$$

3. PROOF OF LOCAL EXISTENCE

Let $u_N(x, t)$ be the solution of problem (2). We first consider the case $\delta > 0$ and $t > 0$. The other cases are treated in Section 4.

LEMMA 3.1. *There exist constants $T_1 > 0$ and k_i, $i = 0, 1, 2, 3$ independent of h but dependent on $f(x)$ and its derivatives of order there and lower, such that*

$$\| u_N(\cdot, t)\|_h \leqslant k_0, \qquad \textit{all } t \tag{3.1}$$

$$| u_N(x_r, t)| \leqslant k_1, \qquad \textit{all } r, \quad 0 \leqslant t \leqslant T_1 \tag{3.2}$$

$$\| D_+D_-^2 u_N(\cdot, t)\|_h \leqslant k_2, \qquad 0 \leqslant t \leqslant T_1 \tag{3.3}$$

and finally with

$$v_N(x, t) = \partial u_N(x, t)/\partial t$$

$$\| v_N(\cdot, t)\|_h \leqslant k_3 \qquad 0 \leqslant t \leqslant T_1 \tag{3.4}$$

Proof.

$$\begin{aligned}(u_N, \partial u_N/\partial t)_h &= \tfrac{1}{2}\partial \| u_N \|_h^2/\partial t\\ &= \tfrac{1}{3}[(u_N, u_N D_0 u_N)_h + (u_N, D_0 u_N^2)_h] + \delta(u_N, D_+D_-^2 u_N)_h \leqslant 0\end{aligned}$$

due to Lemma 2.4. Hence

$$\| u_N(\cdot, t)\|_h^2 \leqslant \| u_N(\cdot, 0)\|_h^2 = \| f \|_h^2 \leqslant 2 \int_0^1 f(x)^2\, dx = k_0^2$$

if $h < h_0$ (this condition is assumed to be fulfilled from now on). So (3.1) is established.

From the triangular inequality and Lemma 2.3 we get

$$\begin{aligned}|\delta|\,\|D_+D_-^2u_N\|_h &\leqslant \|v_N\|_h + \|u_ND_0u_N + D_0u_N^2\|_h/3\\ &\leqslant \|v_N\|_h + \|u_N\|_h \max|D_0u_N|\\ &\leqslant \|v_N\|_h + k_0(\epsilon\|D_+D_-^2u_N\|_h + c(\epsilon)\,k_0).\end{aligned}$$

The choice $\epsilon = |\delta|/2k_0$ gives

$$\|D_+D_-^2u_N\|_h \leqslant 2(\|v_N\|_h + k_0^2c(\epsilon))/|\delta| \leqslant \gamma_1\|v_N\|_h + \gamma_2, \tag{3.5}$$

where γ_1 and γ_2 are independent of h. The function $v_N(x, t)$ satisfies

$$\begin{aligned}\partial v_N(x_r, t)/\partial t = \tfrac{1}{3}[&v_N(x_r, t)\,D_0u_N(x_r, t) + u_N(x_r, t)\,D_0v_N(x_r, t)\\ &+ 2D_0u_N(x_r, t)\,v_N(x_r, t)] + \delta D_+D_-^2v_N(x_r, t).\end{aligned}$$

Multiplication of this equation by $v_N(x_r, t)$ and summation give

$$\begin{aligned}\tfrac{1}{2}\partial\|v_N\|_h^2/\partial t = \tfrac{1}{3}[&(v_N^2, D_0u_N)_h + (v_N, u_ND_0v_N)_h + 2(v_N, D_0u_Nv_N)_h]\\ &+ \delta(v_N, D_+D_-^2v_N)_h,\end{aligned}$$

and Lemma 2.4 gives

$$\begin{aligned}\tfrac{1}{2}\partial\|v_N\|_h^2/\partial t &\leqslant \tfrac{1}{3}|(v_N^2, D_0u_N)_h + (v_N, D_0u_Nv_N)_h|\\ &= \tfrac{1}{3}|(v_N^2, D_0u_N)_h + 0.5h(v_N, D_+u_N\cdot D_0v_N)_h\\ &\qquad + 0.5(v_N, v_N(x-h, t)\,D_0u_N)_h|.\end{aligned}$$

Hence

$$\begin{aligned}\tfrac{1}{2}\partial\|v_N\|_h^2/\partial t &\leqslant \tfrac{1}{3}[1.5\max|D_0u_N|\,\|v_N\|_h^2 + 0.5\max|D_+u_N|\,\|v_N\|_h^2]\\ &\leqslant \tfrac{2}{3}(\epsilon\|D_+D_-^2u_N\|_h + c(\epsilon)\,k_0)\|v_N\|_h^2,\end{aligned}$$

where we have used the Cauchy-Schwarz inequality and Lemma 2.3. If (3.5) is substituted, we get

$$\partial\|v_N\|_h^2/\partial t \leqslant \gamma_3\|v_N\|_h^3 + \gamma_4\|v_N\|_h^2,$$

where γ_3 and γ_4 are constants independent of h.

It is now clear that

$$0 \leqslant \|v_N\|_h^2 \leqslant y,$$

where y is the solution of the initial-value problem

$$dy/dt = \gamma_3y^{3/2} + \gamma_4y, \qquad y(0) = y_0 \geqslant \|v_N(\cdot, 0)\|_h^2. \tag{3.6}$$

The solution y of (3.6) is finite only for

$$t < t_\infty = \frac{2}{\gamma_4} \log\left(1 + \frac{\gamma_4}{\gamma_3 y_0}\right).$$

There is no possibility of choosing constants such that $t_\infty = \infty$. However, we can find a constant k_3 such that (3.4) holds for $0 \leqslant t \leqslant T_1 = t_\infty/2$. The estimate (3.3) then follows from (3.5) and (3.2) is a consequence of Lemma 2.3 and (3.3). This concludes the proof of Lemma 3.1.

With the help of this lemma we can prove that problem (1) has a solution $u(x, t)$ in $0 \leqslant t \leqslant T_1$. We denote by $\phi_N(x, t)$ the discrete Fourier series of $u_N(x, t)$. From Lemma 3.1 and Lemma 2.2 we immediately get

$$\| \partial\phi_N(\cdot\,, t)/\partial t \|^2 \leqslant c_1, \qquad 0 \leqslant t \leqslant T_1, \tag{3.7}$$

$$\| \partial^3\phi_N(\cdot\,, t)/\partial x^3 \|^2 \leqslant c_2, \qquad 0 \leqslant t \leqslant T_1, \tag{3.8}$$

where c_1 and c_2 are constants depending on $f(x)$ but not on h. The rest of the proof is standard. Inequalities (3.7) and (3.8) imply that the sequence $\{\phi_N\}$ is bounded and equicontinuous (in both x and t). The Arzela-Ascoli selection theorem then guarantees the existence of a subsequence $\{\phi_N\}$ which converges to some function $u(x, t)$. It is easy to see that $\partial^3\phi_N/\partial x^3$ converges in the L_2-sense to $\partial^3 u/\partial x^3$. It is also easy to see that $u(x, t)$ is a solution of problem (1) because of the very definition of $\phi_N(x, t)$.

4. Proof of Global Existence

Lemma 4.1. *Let $u(x, t)$ be a solution of problem* (1). *Then there exist a constant β such that*

$$\max | u_x(x, t)| \leqslant \beta \tag{4.1}$$

$$\| v \|^2 \leqslant e^{\beta t} \| ff' + \delta f''' \|^2, \qquad v = \partial u/\partial t. \tag{4.2}$$

Proof. (2.7), (2.8), and Lemma 2.1 give

$$\begin{aligned} | \delta | \, \| u_x \|^2 &\leqslant \int_0^1 | u |^3/3 \, dx + | \alpha_2 | \\ &\leqslant \tfrac{1}{3}(\epsilon \| u_x \| + c(\epsilon) \| u \|) \, \| u \|^2 + | \alpha_2 | \\ &= \tfrac{1}{3}(\epsilon \| u_x \| + c(\epsilon) \sqrt{\alpha_1}) \, \alpha_1 + | \alpha_2 |, \end{aligned}$$

which gives

$$(| \delta |^{1/2} \| u_x \| - \alpha_1 \epsilon | \delta |^{-1/2}/6)^2 \leqslant | \alpha_2 | + c(\epsilon) \, \alpha_1^{1.5}/3 + \alpha_1{}^2 \epsilon^2 | \, 36\delta \, |^{-1}$$

and

$$\| u_x \| \leqslant \beta_0 .$$

To prove (4.1), we now use the identity (2.9):

$$36\delta^2 \| u_{xx} \|^2/5 \leqslant | \alpha_3 | + \int_0^1 u^4 \, dx + 12 \left| \delta \int_0^1 u u_x^2 \, dx \right|$$

$$\leqslant | \alpha_3 | + \max | u |^2 \| u \|^2 + 12 | \delta | \max | u | \, \| u_x \|^2.$$

Lemma 2.1 gives estimates of $\max | u |^2$ and $\max | u |$ in terms of $\| u \|$ and $\| u_x \|$, and hence we get $\| u_{xx} \| \leqslant \beta_1$. Another use of Lemma 2.1 then gives (4.1).

The function $v(x, t) = \partial u(x, t)/\partial t$ satisfies

$$\partial v/\partial t = v u_x + u v_x + \delta v_{xxx} . \tag{4.3}$$

Multiplication of (4.3) by v and integration give

$$\left(v, \frac{\partial v}{\partial t}\right) = \frac{1}{2} \cdot \frac{\partial}{\partial t} \| v \|^2 = (v, v u_x) + (v, u v_x) + \delta (v, v_{xxx}). \tag{4.4}$$

All functions in (4.4) are 1-periodic and, after integration by part, (4.4) is reduced to $\partial \| v \|^2/\partial t = (v^2, u_x)$. (4.1) now gives

$$| \partial \| v \|^2/\partial t | \leqslant \max | u_x | \, \| v \|^2 \leqslant \beta \| v \|^2,$$

which implies

$$\| v \|^2 = \| \partial u(\cdot , t)/\partial t \|^2 \leqslant e^{\beta t} \| \partial u(\cdot , 0)/\partial t \|^2$$

$$= e^{\beta t} \| f(x) f'(x) + \delta f'''(x) \|^2,$$

i.e. (4.2).

The function $\partial u(x, 0)/\partial t$ can be expressed in terms of the derivatives $d^n f(x)/dx^n$, $n = 0, 1, 2, 3$, and hence $\| \partial u(\cdot , 0)/\partial t \|$ is bounded. From (1.1) and the triangular inequality we get

$$\| \partial^3 u/\partial x^3 \| \leqslant b_1 \| v \| + b_2 . \tag{4.5}$$

The bound (4.2) guarantees that $\partial u/\partial t$ is square integrable for every t and (4.5) that problem (1) with initial function $u(x, T_1)$ instead of $f(x)$ has a solution for $T_1 \leqslant t \leqslant T_2$. Thus we get a solution for $0 \leqslant t \leqslant T$. We will show that by repeating the extension procedure we can obtain existence for all $t > 0$. For this purpose suppose that existence can be proven only in $0 \leqslant t < T < \infty$. If we look at the expression for t_∞ in Section 3, we find that only y_0 depends on t. But y_0 can, because of (4.2), be chosen to hold in the whole interval $0 \leqslant t < T$. Consequently, if we consider problem (1) with $f(x) = u(x, \tau)$ for some τ sufficiently close to T, we can, by using the local procedure, get existence for values of t lying outside $0 \leqslant t \leqslant T$. As this result contradicts the assumption we have proven existence in the upper half-plane for the case $\delta > 0$.

To prove existence in the lower half-plane we consider the equation

$$u_t = -uu_x - \delta u_{xxx}, \tag{4.6}$$

which arises when making the transformation $(1 - x) \to x$. If (4.6) is substituted for (1.1) in problem (1) we can prove existence in the lower half-plane. This is possible because then we get an interval $T_1 \leqslant t \leqslant 0$ in Lemma 3.1 (we still assume $\delta > 0$). Thus we have existence in the whole plane for $\delta > 0$. If $\delta < 0$ we first get existence in the lower half-plane and then in the upper. This concludes the proof of the existence part of Theorem 1.

5. Uniqueness

Assume that $u(x, t)$ and $v(x, t)$ both are solutions of problem (1). The function $w = u - v$ satisfies

$$\begin{cases} w_t = uu_x - vv_x + \delta w_{xxx} = wu_x + vw_x + \delta w_{xxx} \\ w(x, 0) = 0 \\ w(x, t) = w(x + 1, t). \end{cases}$$

Hence

$$\begin{aligned} (w, w_t) &= \tfrac{1}{2}\partial \| w \|^2/\partial t = (w, wu_x) + (w, vw_x) + \delta(w, w_{xxx}) \\ &= (w^2, u_x) - (w^2, v_x)/2 \end{aligned}$$

by periodicity.

The estimate (4.1) now gives

$$\partial \| w \|^2/\partial t \leqslant 1.5\beta \| w \|^2.$$

As $\| w(\cdot, 0)\|^2 = 0$, it is clear that $\| w(\cdot, t)\|^2 = 0$ for all t which implies uniqueness.

This terminates the proof of Theorem 1.

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